

# GROUPS WITH FAITHFUL IRREDUCIBLE PROJECTIVE UNITARY REPRESENTATIONS

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**ABSTRACT.** For a countable group  $\Gamma$  and a multiplier  $\zeta \in Z^2(\Gamma, \mathbf{T})$ , we study the property of  $\Gamma$  having a unitary projective  $\zeta$ -representation which is both irreducible and projectively faithful. Theorem 1 shows that this property is equivalent to  $\Gamma$  being the quotient of an appropriate group by its centre. Theorem 4 gives a criterion in terms of the minisocle of  $\Gamma$ . Several examples are described to show the existence of various behaviours.

## 1. Introduction

For a Hilbert space  $\mathcal{H}$ , we denote by  $\mathcal{U}(\mathcal{H})$  the group of its unitary operators. We identify  $\mathbf{T} := \{z \in \mathbf{C} \mid |z| = 1\}$  with the centre of  $\mathcal{U}(\mathcal{H})$ , namely with the scalar multiples of the identity operator  $\text{id}_{\mathcal{H}}$ , we denote by  $\mathcal{PU}(\mathcal{H}) := \mathcal{U}(\mathcal{H})/\mathbf{T}$  the *projective unitary group* of  $\mathcal{H}$ , and by

$$p_{\mathcal{H}} : \mathcal{U}(\mathcal{H}) \longrightarrow \mathcal{PU}(\mathcal{H})$$

the canonical projection.

Let  $\Gamma$  be a group. A projective unitary representation, or shortly here a *projective representation*, of  $\Gamma$  in  $\mathcal{H}$  is a mapping

$$\pi : \Gamma \longrightarrow \mathcal{U}(\mathcal{H})$$

such that  $\pi(e) = \text{id}_{\mathcal{H}}$  and such that the composition

$$\underline{\pi} := p_{\mathcal{H}}\pi : \Gamma \longrightarrow \mathcal{PU}(\mathcal{H})$$

is a homomorphism of groups. When we find it useful, we write  $\mathcal{H}_{\pi}$  for the Hilbert space of a projective representation  $\pi$ .

A projective representation  $\pi$  of a group  $\Gamma$  is *projectively faithful*, or shortly *P-faithful*, if the corresponding homomorphism  $\underline{\pi}$  is injective.

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The *projective kernel* of  $\pi$  is the normal subgroup

$$(1) \quad \text{Pker}(\pi) = \ker(\underline{\pi}) = \{x \in \Gamma \mid \pi(x) \in \mathbf{T}\}$$

of  $\Gamma$ . In case  $\pi$  is a unitary representation,  $\ker(\pi)$  is a subgroup of  $\text{Pker}(\pi)$ , sometimes called the *quasikernel* of  $\pi$ , which can be a proper subgroup, so that faithfulness of  $\pi$  does not imply P-faithfulness.

A projective representation  $\pi$  is *irreducible* if the only closed  $\pi(\Gamma)$ -invariant subspace of  $\mathcal{H}_\pi$  are  $\{0\}$  and  $\mathcal{H}_\pi$ .

As a continuation of [BeHa–08], the present paper results of our effort to understand which groups have irreducible P-faithful projective representations. Our first observation is a version in the present context of Satz 4.1 of [Pahl–68]. We denote by  $Z(\Gamma)$  the centre of a group  $\Gamma$ .

**Theorem 1.** *For a group  $\Gamma$ , the following two properties are equivalent:*

- (i)  $\Gamma$  affords an irreducible P-faithful projective representation;
- (ii) there exists a group  $\Delta$  which affords an irreducible faithful unitary representation and which is such that  $\Delta/Z(\Delta) \approx \Gamma$ .

*If, moreover,  $\Gamma$  is countable, these properties are also equivalent to:*

- (iii) there exists a countable group  $\Delta$  as in (ii).

Countable groups which have irreducible faithful unitary representations have been characterised in [BeHa–08], building up on results of [Gasc–54] for finite groups.

A group  $\Gamma$  is *capable* if there exists a group  $\Delta$  with  $\Gamma \approx \Delta/Z(\Delta)$ , and *incapable* otherwise. The notion appears in [Baer–38], which contains a criterion of capability for abelian groups which are direct sums of cyclic groups (for this, see also [BeFS–79]), and the terminology “capable” is that of [HaSe–64]. Conditions for capability (several of them being either necessary or sufficient) are given in Chapter IV of [BeTa–82].

The *epicentre* of a group  $\Gamma$  is the largest central subgroup  $A$  such that the quotient projection  $\Gamma \rightarrow \Gamma/A$  induces in homology an injective homomorphism  $H_2(\Gamma, \mathbf{Z}) \rightarrow H_2(\Gamma/A, \mathbf{Z})$ , where  $\mathbf{Z}$  is viewed as a trivial module. This group was introduced in [BeFS–79] and [BeTa–82], with a formally different definition; the terminology is from [Elli–98], and the characterisation given above appears in Theorem 4.2 of [BeFS–79].

**Proposition 2** (Beyl-Felgner-Schmid-Ellis). *Let  $\Gamma$  be a group and let  $Z^*(\Gamma)$  denote its epicentre.*

- (i)  $\Gamma$  is capable if and only if  $Z^*(\Gamma) = \{e\}$ .
- (ii)  $\Gamma/Z^*(\Gamma)$  is capable in all cases.
- (iii) A perfect group with non trivial centre is incapable.

**Corollary 3.** *A perfect group with non-trivial centre has no P-faithful projective representation.*

A *multiplier* on  $\Gamma$  is a mapping  $\zeta : \Gamma \times \Gamma \longrightarrow \mathbf{T}$  such that

$$(2) \quad \zeta(e, x) = \zeta(x, e) = 1 \quad \text{and} \quad \zeta(x, y)\zeta(xy, z) = \zeta(x, yz)\zeta(y, z)$$

for all  $x, y, z \in \Gamma$ . We denote by  $Z^2(\Gamma, \mathbf{T})$  the set of all these, which is an abelian group for the pointwise product. A projective representation  $\pi$  of  $\Gamma$  in  $\mathcal{H}$  determines a unique multiplier  $\zeta_\pi = \zeta$  such that

$$(3) \quad \pi(x)\pi(y) = \zeta(x, y)\pi(xy)$$

for all  $x, y \in \Gamma$ ; we say then that  $\pi$  is a  $\zeta$ -*representation* of  $\Gamma$ . Conversely, any  $\zeta \in Z^2(\Gamma, \mathbf{T})$  occurs in such a way; indeed,  $\zeta$  is the multiplier determined by the *twisted left regular  $\zeta$ -representation*  $\lambda_\zeta$  of  $\Gamma$ , defined on the Hilbert space  $\ell^2(\Gamma)$  by

$$(4) \quad (\lambda_\zeta(x)\varphi)(y) = \zeta(x, x^{-1}y)\varphi(x^{-1}y).$$

A good reference for these regular  $\zeta$ -representations is [Klep-62]. In the special case  $\zeta = 1$ , a  $\zeta$ -representation of  $\Gamma$  is just a *unitary representation* of  $\Gamma$ ; but we repeat that

**First standing assumption.** In this paper, by “representation”, we always mean “unitary representation”.

For a projective representation of  $\Gamma$  in  $\mathbf{C}$ , namely for a mapping  $\nu : \Gamma \longrightarrow \mathbf{T}$  with  $\nu(e) = 1$ , let  $\zeta_\nu \in Z^2(\Gamma, \mathbf{T})$  denote the corresponding multiplier, namely the mapping defined by

$$(5) \quad \zeta_\nu(x, y) = \nu(x)\nu(y)\nu(xy)^{-1}.$$

We denote by  $B^2(\Gamma, \mathbf{T})$  the set of all multipliers of the form  $\zeta_\nu$ , which is a subgroup of  $Z^2(\Gamma, \mathbf{T})$ , and by  $H^2(\Gamma, \mathbf{T}) := Z^2(\Gamma, \mathbf{T})/B^2(\Gamma, \mathbf{T})$  the quotient group; as usual,  $\zeta, \zeta' \in Z^2(\Gamma, \mathbf{T})$  are *cohomologous* if they have the same image in  $H^2(\Gamma, \mathbf{T})$ .

Given a  $\zeta$ -representation  $\pi$  of  $\Gamma$  in  $\mathcal{H}$ , there is a standard bijection between:

- the set of projective representations  $\pi' : \Gamma \longrightarrow \mathcal{U}(\mathcal{H})$  such that  $p_{\mathcal{H}}\pi = p_{\mathcal{H}}\pi'$ , on the one hand,
- and the set of multipliers cohomologous to  $\zeta$ , on the other hand.

In other terms, a group homomorphism  $\underline{\pi}$  of  $\Gamma$  in  $\mathcal{PU}(\mathcal{H})$  determines a class<sup>1</sup> in  $H^2(\Gamma, \mathbf{T})$ , and the set of projective representations covering  $\underline{\pi}$  is in bijection with the representatives of this class in  $Z^2(\Gamma, \mathbf{T})$ . Observe that  $\pi$  and  $\pi'$  above are together irreducible or not, and together P-faithful or not.

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<sup>1</sup>This is of course the class associated to the extension of  $\Gamma$  by  $\mathbf{T}$  pulled back by  $\underline{\pi}$  of the extension  $\{1\} \longrightarrow \mathbf{T} \longrightarrow \mathcal{U}(\mathcal{H}) \longrightarrow \mathcal{PU}(\mathcal{H}) \longrightarrow \{1\}$ . See for example Section 6.6 in [Weib-94].

For much more on projective representations and multipliers, in the setting of separable locally compact groups, see [Mack–58]; for a very short but informative exposition on earlier work, starting with that of Schur, see [Kall–84]. Note that other authors (such as Kleppner) use “multiplier representation” and “projective representation” when Mackey uses “projective representation” and “homomorphism in  $\mathcal{PU}(\mathcal{H})$ ”, respectively.

**Definition.** *Given a group  $\Gamma$  and a multiplier  $\zeta \in Z^2(\Gamma, \mathbf{T})$ , the group  $\Gamma$  is irreducibly  $\zeta$ -represented if it has an irreducible  $P$ -faithful  $\zeta$ -representation.*

This depends only on the class  $\underline{\zeta} \in H^2(\Gamma, \mathbf{T})$  of  $\zeta$ .

For a group  $\Gamma$ , recall that a *foot* of  $\Gamma$  is a minimal normal subgroup, that the *minisocle* is the subgroup  $MS(\Gamma)$  of  $\Gamma$  generated by the union of all finite feet of  $\Gamma$ , and that  $MA(\Gamma)$  is the subgroup of  $MS(\Gamma)$  generated by the union of all finite abelian feet of  $\Gamma$ . It is obvious that  $MS(\Gamma)$  and  $MA(\Gamma)$  are characteristic subgroups of  $\Gamma$ ; it is easy to show that  $MA(\Gamma)$  is abelian and is a direct factor of  $MS(\Gamma)$ . For all this, we refer to Proposition 1 in [BeHa–08].

Let  $N$  be a normal subgroup of  $\Gamma$  and  $\sigma$  a  $\zeta$ -representation of  $N$ , for some  $\zeta \in Z^2(N, \mathbf{T})$ . If  $\zeta = 1$  (the case of ordinary representations), define the  $\Gamma$ -kernel of  $\sigma$  by

$$\ker_{\Gamma}(\sigma) = \ker \left( \bigoplus_{\gamma \in \Gamma} \sigma^{\gamma} \right)$$

where  $\sigma^{\gamma}(x) := \sigma(\gamma x \gamma^{-1})$ ; say, as in [BeHa–08], that  $\sigma$  is  $\Gamma$ -faithful if this  $\Gamma$ -kernel is reduced to  $\{e\}$ ; when  $\zeta$  is the restriction to  $N$  of a multiplier (usually denoted by  $\zeta$  again) in  $Z^2(\Gamma, \mathbf{T})$ , there is an analogous notion for the general case ( $\zeta \neq 1$ ), called  $\Gamma$ - $P$ -faithfulness, used in Theorem 4, but defined only in Section 3 below. Before the next result, we find it convenient to define one more property.

**Definition.** *A group  $\Gamma$  has Property (Fab) if any normal subgroup of  $\Gamma$  generated by one conjugacy class has a finite abelianisation.*

Examples of groups which enjoy Property (Fab) include finite groups,  $SL_n(\mathbf{Z})$  for  $n \geq 3$ , and more generally lattices in a finite product  $\prod_{\alpha \in A} G_{\alpha}$  of simple groups  $G_{\alpha}$  over (possibly different) local fields  $k_{\alpha}$  when  $\sum_{\alpha \in A} k_{\alpha} - \text{rank}(G_{\alpha}) \geq 2$  (see [Marg–91], IV.4.10, and Example VI below). They also include abelian locally finite groups, and more generally torsion groups which are FC, namely which are such that all their conjugacy classes are finite; in particular, they include groups of the form  $MS(\Gamma)$  and  $MA(\Gamma)$ .

**Theorem 4.** *Let  $\Gamma$  be a countable group and let  $\zeta \in Z^2(\Gamma, \mathbf{T})$ . Consider the following conditions:*

- (i)  $\Gamma$  is irreducibly  $\zeta$ -represented;
- (ii)  $MS(\Gamma)$  has a  $\Gamma$ - $P$ -faithful irreducible  $\zeta$ -representation;
- (iii)  $MA(\Gamma)$  has a  $\Gamma$ - $P$ -faithful irreducible  $\zeta$ -representation.

*Then (i)  $\implies$  (ii)  $\iff$  (iii).*

*If, moreover,  $\Gamma$  has Property (Fab), then (ii)  $\implies$  (i), so that (i), (ii), and (iii) are equivalent.*

The hypothesis “ $\Gamma$  countable” is essential because our arguments use measure theory and direct integrals; in fact, Theorem 4 fails in general for uncountable groups (see Example (VII) in [BeHa–08], Page 863). About the converse of (ii)  $\implies$  (i), see Example I below.

Recall that a group  $\Gamma$  has *infinite conjugacy classes*, or is *icc*, if  $\Gamma \neq \{e\}$  and if any conjugacy class in  $\Gamma \setminus \{e\}$  is infinite. For example, a lattice in a centreless connected semisimple Lie group without compact factors is icc, as a consequence of Borel Density Theorem (see Example VI).

**Corollary 5.** *Let  $\Gamma$  be a countable group which has Property (Fab) and which fulfills at least one of the three following conditions:*

- (i)  $\Gamma$  is torsion free;
- (ii)  $\Gamma$  is icc;
- (iii)  $\Gamma$  has a faithful primitive action on an infinite set.

*Then, for any  $\zeta \in Z^2(\Gamma, \mathbf{T})$ , the group  $\Gamma$  is irreducibly  $\zeta$ -represented.*

Indeed, any of Conditions (i) to (iii) implies that  $MS(\Gamma) = \{e\}$ . Recall that, if  $\Gamma$  fulfills (iii) on an infinite set  $X$ , any normal subgroup  $N \neq \{e\}$  acts transitively on  $X$ , and therefore is infinite (see [GeGl–08]).

A group can be either irreducibly represented or not, and also either irreducibly  $\zeta$ -represented or not (for some  $\zeta$ ). These dichotomies separate groups in four classes, each one illustrated in Section 2 by one of Examples I to IV below. Examples V and VI illustrate the same class as Example I.

Section 3 contains standard material on multipliers, and the definition of  $\Gamma$ - $P$ -faithfulness; mind the “second standing assumption” on the normalisation of multipliers, which applies to all other sections. In Section 4, we review central extensions and prove Theorem 1. Sections 5 and 6 contain the proof of Theorem 4, respectively the part which involves our “Property (Fab)” and the part which does not.

Section 7 contains material on (in)capability, and the proof of Proposition 2. Section 8 describes a construction of irreducible P-faithful projective representation of a class of abelian groups, and expands on Example II. The last section is a digression to point out a fact from homological algebra which in our opinion is not quoted often enough in the literature on projective representations.

## 2. Examples

**Example I.** The implication (ii)  $\implies$  (i) of Theorem 4 does not hold for a free abelian group  $\mathbf{Z}^n$  ( $n \geq 1$ ) and the unit multiplier<sup>2</sup>  $\zeta = 1 \in Z^2(\mathbf{Z}^n, \mathbf{T})$ . Indeed, on the one hand, Condition (ii) of Theorem 4 is satisfied since  $MS(\mathbf{Z}^n) = \{0\}$ . On the other hand, since  $\mathbf{Z}^n$  is abelian, any irreducible  $\zeta$ -representation (that is any ordinary irreducible representation) is one-dimensional, so that its projective kernel is the whole of  $\mathbf{Z}^n$ , and therefore  $\mathbf{Z}^n$  is *not* irreducibly  $\zeta$ -represented. Moreover,  $\mathbf{Z}^n$  being for any  $n \geq 1$  a (dense) subgroup of  $\mathbf{T}$ , it has an irreducible faithful representation of dimension one.

**Example II.** There are groups which do not afford any irreducible faithful representation but which do have projective representations which are irreducible and P-faithful.

The Vierergruppe  $\mathbf{V} = \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ , being finite abelian non-cyclic, does not have any irreducible faithful representation. If  $\zeta \in Z^2(\mathbf{V}, \mathbf{T})$  is a cocycle representing the non-trivial cohomology class in  $H^2(\mathbf{V}, \mathbf{T}) \approx \mathbf{Z}/2\mathbf{Z}$ , then  $\mathbf{V}$  has a  $\zeta$ -representation of degree 2 which is both irreducible and P-faithful, essentially given by the Pauli matrices (see Section IV.3 in [Simo–96]).

Part of this carries over to any non-trivial finite abelian group of the form  $L \times L$ . More on this in Section 8.

**Example III.** Let us first recall a few basic general facts about irreducible projective representations of a finite group  $\Gamma$ . The cohomology group  $H^2(\Gamma, \mathbf{T})$  is isomorphic to the homology group  $H_2(\Gamma, \mathbf{Z})$ , and is finite. Choose a multiplier  $\zeta \in Z^2(\Gamma, \mathbf{T})$ , say normalised (see Section 3 below). An element  $x \in \Gamma$  is  $\zeta$ -regular if  $\zeta(x, y) = \zeta(y, x)$  whenever  $y \in \Gamma$  commutes with  $x$ ; it can be checked that a conjugate of a regular element is again regular. Let  $h(\zeta)$  denote the number of conjugacy classes of  $\zeta$ -regular elements in  $\Gamma$ . Then it is known that  $\Gamma$  has exactly  $h(\zeta)$  irreducible  $\zeta$ -representations, up to unitary equivalence, say

<sup>2</sup> Recall that  $H^2(\mathbf{Z}, \mathbf{T}) = \{0\}$ , because  $\mathbf{Z}$  is free, so that

$$Z^2(\mathbf{Z}, \mathbf{T}) = B^2(\mathbf{Z}, \mathbf{T}) = \text{Mapp}(\mathbf{Z}, \mathbf{T}) / \text{Hom}(\mathbf{Z}, \mathbf{T}).$$

Also  $H^2(\mathbf{Z}^n, \mathbf{T}) = \mathbf{Z}^{n(n-1)/2}$  for all  $n \geq 1$ .

of degrees  $d_1, \dots, d_{h(\zeta)}$ ; moreover each  $d_j$  divides the order of  $\Gamma$ , and  $\sum_{j=1}^{h(\zeta)} d_j^2 = |\Gamma|$ . See Chapter 6 in [BeZh–98], in particular Corollary 10 and Theorem 13 Page 149.

Clearly  $h(\zeta) \leq h(1)$  for all  $\zeta \in Z^2(\Gamma, \mathbf{T})$ . It follows from Lemma 11 below that, if  $\zeta \neq 1$ , then  $d_j \geq 2$  for all  $j \in \{1, \dots, h(\zeta)\}$ .

Now, for the gist of this Example III, assume that  $\Gamma$  is a nonabelian finite simple group. Then, except for the unit character, any representation of  $\Gamma$  is faithful and any projective representation of  $\Gamma$  is P-faithful.

**Example IV.** Let  $\Gamma$  be a perfect group. Its *universal central extension*  $\tilde{\Gamma}$  is a perfect group with centre the<sup>3</sup> *Schur multiplier*  $H_2(\Gamma, \mathbf{Z})$  and central quotient  $\Gamma$  [Kerv–70]. If this Schur multiplier is not  $\{0\}$ ,  $\tilde{\Gamma}$  is incapable, and therefore does not have any irreducible P-faithful projective representation (Corollary 3). If  $\Gamma$  is as in (i) or (ii) below,  $\tilde{\Gamma}$  is moreover not irreducibly represented (by [Gasc–54] and [BeHa–08]):

- (i)  $\Gamma$  is a finite simple group with  $H_2(\Gamma, \mathbf{Z})$  not cyclic<sup>4</sup>. The complete list of such groups is given in Theorem 4.236, Page 301 of [Gore–82], and includes the finite simple group  $\mathrm{PSL}_3(\mathbf{F}_4)$ , also denoted by  $A_2(4)$ , one of the two finite simple groups of order 20160.
- (ii)  $\Gamma$  is one of the Steinberg groups  $\mathrm{St}_3(\mathbf{Z})$  and  $\mathrm{St}_4(\mathbf{Z})$ , which are the universal central extensions of  $\mathrm{SL}_3(\mathbf{Z})$  and  $\mathrm{SL}_4(\mathbf{Z})$ , respectively. Indeed, van der Kallen<sup>5</sup> [Kall–74] has shown that

$$H_2(\mathrm{SL}_3(\mathbf{Z}), \mathbf{Z}) \approx H_2(\mathrm{SL}_4(\mathbf{Z}), \mathbf{Z}) \approx \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}.$$

Thus, these groups are not irreducibly represented by [BeHa–08]. (For  $n \geq 5$ , it is known that  $H_2(\mathrm{SL}_n(\mathbf{Z}), \mathbf{Z}) \approx \mathbf{Z}/2\mathbf{Z}$ ; see [Miln–71], Page 48. And  $H_2(\mathrm{SL}_2(\mathbf{Z}), \mathbf{Z}) = \{0\}$ , see the comments after Proposition 18.)

**Example V.** This example and the next one will show, besides the  $\mathbf{Z}^n$ ’s of Example I, groups which are irreducibly represented, but which do not have any irreducible P-faithful representation.

Any finite perfect group  $\Gamma$  with centre  $Z(\Gamma)$  cyclic and not  $\{0\}$  has these properties, by Gaschütz theorem and by Corollary 3. This is for example the case of the quasi simple group  $\mathrm{SL}_n(\mathbf{F}_q)$  whenever the finite

<sup>3</sup>Recall that, for a perfect group  $\Gamma$  and a trivial  $\Gamma$ -module  $A$ , we have  $H^2(\Gamma, A) \approx \mathrm{Hom}(H_2(\Gamma, \mathbf{Z}), A)$  as a consequence of the universal coefficient theorem for cohomology; in particular,  $H^2(\Gamma, \mathbf{T})$  is the Pontryagin dual of the Schur multiplier. In case  $H_2(\Gamma, \mathbf{Z})$  is moreover finite, e.g. if  $\Gamma$  is perfect and finite,  $H^2(\Gamma, \mathbf{T})$  is isomorphic to the Schur multiplier (non-canonically).

<sup>4</sup>If  $H_2(\Gamma, \mathbf{Z})$  is cyclic not  $\{0\}$ , see Example V.

<sup>5</sup>Thanks to Andrei Rapinchuk for this reference.

field  $\mathbf{F}_q$  has non-trivial  $n$ th roots of unity, so that  $Z(\mathrm{SL}_n(\mathbf{F}_q)) = \{\lambda \in \mathbf{F}_q \mid \lambda^n = 1\}$  is cyclic and not  $\{e\}$ . (As usual,  $\mathrm{SL}_2(\mathbf{F}_2)$  and  $\mathrm{SL}_2(\mathbf{F}_3)$  are ruled out.)

The groups  $\mathrm{SL}_{2n}(\mathbf{Z})$ , for  $2n \geq 4$ , are perfect with centre cyclic of order 2, and therefore incapable, so that Corollary 3 applies; the group  $\mathrm{SL}_2(\mathbf{Z})$ , which is not perfect, is also incapable (see Section 7). On the other hand, since for all  $n \geq 1$  the minisocle of  $\mathrm{SL}_{2n}(\mathbf{Z})$  coincides with its centre, of order 2, these groups do have representations which are irreducible and faithful. These considerations hold also for the symplectic groups  $\mathrm{Sp}_{2n}(\mathbf{Z})$ ,  $2n \geq 6$ , which are perfect [Rein–95].

**Example VI.a.** Let  $B$  be a finite set. For  $\beta \in B$ , let  $\mathbf{k}_\beta$  be a local field and  $\mathbf{G}_\beta$  be a nontrivial connected semi-simple group defined over  $\mathbf{k}_\beta$ , without  $\mathbf{k}_\beta$ -anisotropic factor. Set  $G = \prod_{\beta \in B} \mathbf{G}_\beta(\mathbf{k}_\beta)$ , with its locally compact topology which makes it a  $\sigma$ -compact, metrisable, compactly generated group. Let  $\Gamma$  be an irreducible lattice in  $G$ .

If  $N$  is a finite normal subgroup of  $\Gamma$ , we claim that  $N$  is central in  $\Gamma$ . If there are several factors ( $|B| \geq 2$ ), the claim is a consequence of the fact that the projection of the lattice in each factor is dense, by irreducibility. If  $|B| = 1$ , consider  $x \in N$ . The centraliser  $Z_\Gamma(x)$  of  $x$  in  $\Gamma$  is also a lattice in  $G$  because it is of finite index in  $\Gamma$ . By the Borel-Wang density theorem (Corollary 4.4 of Chapter II in [Marg–91]),  $Z_\Gamma(x)$  is Zariski-dense in  $G$ , so that  $x$  commutes with every element of  $G$ , and this proves the claim.

It follows that, if moreover the centre of  $G$  is finite cyclic, then  $MS(\Gamma) = MA(\Gamma)$  is also a finite cyclic group, so that  $\Gamma$  is irreducibly represented by [BeHa–08].

**Example VI.b.** To continue this same example, let us particularise the situation to the case of a non-compact semi-simple real Lie group  $G$ , which is connected, not simply connected, and with a non-trivial centre. Let  $\Gamma$  be a lattice in  $G$  with a non-trivial centre  $Z(\Gamma)$ . Denote by  $\tilde{\Gamma}$  the inverse image of  $\Gamma$  in the universal cover of  $G$  and by  $p : \tilde{\Gamma} \rightarrow \Gamma$  the canonical projection. Observe that  $Z(\tilde{\Gamma}) = p^{-1}(Z(\Gamma))$ . Choose a set-theoretical section  $s : \Gamma \rightarrow \tilde{\Gamma}$  for  $p$  with  $s(e) = e$  and a character  $\chi \in \mathrm{Hom}(Z(\tilde{\Gamma}), \mathbf{T})$ . Define a mapping

$$\zeta : \Gamma \times \Gamma \longrightarrow \mathbf{T}, \quad \zeta(x, y) = \chi(s(x)s(y)s(xy)^{-1}).$$

It is straightforward to check that  $\zeta$  is a multiplier, namely that  $\zeta \in Z^2(\Gamma, \mathbf{T})$ .

[Classes of multipliers of this kind are not arbitrary. They correspond precisely to those classes in  $H^2(\Gamma, \mathbf{T})$  which are restrictions of classes in the appropriately defined group  $H^2(G, \mathbf{T})$ . The latter group is known



to be isomorphic to  $\text{Hom}(\pi_1(G), \mathbf{T})$ ; see Proposition 3.4 in [Moor–64] and [BaMi–00].

It is obvious that, if  $\chi$  extends to a unitary character  $\tilde{\chi}$  of  $\tilde{\Gamma}$ , then  $\zeta$  belongs to  $B^2(\Gamma, \mathbf{T})$ . Indeed, in this case  $\zeta = \zeta_\nu$  for  $\nu : \Gamma \rightarrow \mathbf{T}$  defined by  $\nu(x) = \tilde{\chi}(s(x))$ . Conversely, assume that  $\zeta = \zeta_\nu$  for some  $\nu : \Gamma \rightarrow \mathbf{T}$  with  $\nu(e) = 1$ . As in the proof of Theorem 1.1 in [BaMi–00], define  $v : \tilde{\Gamma} \rightarrow \mathbf{T}$  by  $v(x) = \chi(x^{-1}s(p(x)))^{-1}$  for  $x \in \tilde{\Gamma}$  (observe that  $x^{-1}s(p(x)) \in Z(\tilde{\Gamma})$ , so that  $v(x)$  is well-defined). One checks that

$$\zeta(p(x), p(y)) = v(x)v(y)v(xy))^{-1} \quad \forall x, y \in \tilde{\Gamma}.$$

It follows that the function  $\tilde{\chi} : \tilde{\Gamma} \rightarrow \mathbf{T}$ , defined by  $\tilde{\chi}(x) = \nu(p(x))v(x)^{-1}$ , is a character of  $\tilde{\Gamma}$  which extends  $\chi$ .

As a consequence, if the intersection of  $Z(\tilde{\Gamma})$  with the commutator subgroup  $[\tilde{\Gamma}, \tilde{\Gamma}]$  is not reduced to  $\{e\}$ , we can find  $\chi$  such that the corresponding multiplier  $\zeta$  does not belong to  $B^2(\Gamma, \mathbf{T})$ . We provide in VI.c below an example for which this does occur.

We claim that  $\Gamma$  has no P-faithful irreducible  $\zeta$ -representation. By Theorem 4, it suffices to show that  $MS(\Gamma)$  has no  $\Gamma$ -P-faithful irreducible  $\zeta$ -representation.

The group  $MS(\Gamma)$  coincides with  $Z(\Gamma)$  as we have shown in Part VI.a of the present example. Observe that  $s(x) \in Z(\tilde{\Gamma})$  for every  $x \in Z(\Gamma)$ . We have therefore  $\zeta(x, y) = \zeta(y, x)$  for all  $x \in Z(\Gamma)$  and  $y \in \Gamma$ . In particular, it follows that the restriction of  $\zeta$  to  $Z(\Gamma)$  is trivial (see Lemma 7.2 in [Klep–65]). Upon changing  $\zeta$  inside its cohomology class, we can assume that  $\zeta(x, y) = 1$  for all  $x, y \in Z(\Gamma)$ .

Let  $\sigma$  be an irreducible  $\zeta$ -representation of  $Z(\Gamma)$ . Since the restriction of  $\zeta$  to  $Z(\Gamma)$  is trivial, we have  $\text{Pker } \sigma = Z(\Gamma)$ . From the fact that  $\zeta(x, y) = \zeta(y, x)$  for all  $x \in Z(\Gamma)$  and  $y \in \Gamma$ , it follows that  $\text{Pker}_\Gamma \sigma = Z(\Gamma)$ ; see Remark (B) after Proposition 8. Hence,  $\sigma$  is not  $\Gamma$ -P-faithful since  $Z(\Gamma)$  is non-trivial by assumption.

**Example VI.c.** Let  $\Delta$  be the fundamental group of a closed surface of genus 2, viewed as a subgroup of  $PSL_2(\mathbf{R})$ . Let  $\Gamma$  be the inverse image of  $\Delta$  in  $SL_2(\mathbf{R})$ ; observe that  $Z(\Gamma)$  is the two-element group. The group  $\tilde{\Gamma}$ , the discrete subgroup of the universal cover of  $SL_2(\mathbf{R})$  defined in VI.b, has a presentation with (see IV.48 in [Harp–00])

$$\begin{aligned} &\text{generators: } a_1, a_2, b_1, b_2, c \\ &\text{and relations: } c \text{ is central, and } [a_1, b_1][a_2, b_2] = c^2. \end{aligned}$$

In particular, the intersection of  $Z(\tilde{\Gamma})$  with  $[\tilde{\Gamma}, \tilde{\Gamma}]$  is non trivial .

### 3. $\Gamma$ -P-faithfulness for projective representations of normal subgroups of $\Gamma$

Let  $N$  be a normal subgroup of a group  $\Gamma$  and let  $\zeta \in Z^2(N, \mathbf{T})$ . Let  $\sigma$  be a  $\zeta$ -representation of  $N$ .

For  $\gamma \in \Gamma$ , the mapping

$$N \ni x \longmapsto \sigma(\gamma x \gamma^{-1}) \in \mathcal{U}(\mathcal{H}_\sigma)$$

is in general not a  $\zeta$ -representation of  $N$ , but is a  $\zeta^\gamma$ -representation of  $N$ , where  $\zeta^\gamma \in Z^2(N, \mathbf{T})$  is defined by

$$(6) \quad \zeta^\gamma(x, y) = \zeta(\gamma x \gamma^{-1}, \gamma y \gamma^{-1})$$

for all  $x, y \in N$ .

Suppose moreover that  $\zeta$  is the restriction to  $N$  of some multiplier on  $\Gamma$ . Then the multiplier  $\zeta^\gamma$  is cohomologous to  $\zeta$ ; more precisely:

**Lemma 6** (Mackey). *Let  $\zeta \in Z^2(\Gamma, \mathbf{T})$ , let  $\sigma$  be a  $\zeta$ -representation of  $N$ , and let  $\gamma \in \Gamma$ .*

(i) *Define a mapping  $\nu_\gamma : N \longrightarrow \mathbf{T}$  by  $\nu_\gamma(x) = \frac{\zeta(\gamma, x)\zeta(\gamma x, \gamma^{-1})}{\zeta(\gamma^{-1}, \gamma)}$ . Then*

$$(7) \quad \frac{\zeta^\gamma(x, y)}{\zeta(x, y)} = \frac{\nu_\gamma(xy)}{\nu_\gamma(x)\nu_\gamma(y)}$$

for all  $x, y \in N$ . In particular,  $\underline{\zeta^\gamma} = \underline{\zeta} \in H^2(\Gamma, \mathbf{T})$ .

(ii) *Define a mapping  $\sigma^\gamma : N \longrightarrow \mathcal{U}(\mathcal{H}_\sigma)$  by*

$$(8) \quad \sigma^\gamma(x) = \zeta(\gamma, x)\zeta(\gamma x, \gamma^{-1})\sigma(\gamma x \gamma^{-1}).$$

Then

$$(9) \quad \sigma^\gamma(x)\sigma^\gamma(y) = \zeta(\gamma^{-1}, \gamma)\zeta(x, y)\sigma^\gamma(xy)$$

for all  $x, y \in N$ .

*Proof.* For (i), we refer to Lemma 4.2 in [Mack–58], of which the proof uses (2) from Section 1. For (ii), we have

$$\begin{aligned} \sigma^\gamma(x)\sigma^\gamma(y) &= \\ \zeta(\gamma, x)\zeta(\gamma x, \gamma^{-1})\zeta(\gamma, y)\zeta(\gamma y, \gamma^{-1})\zeta(\gamma x \gamma^{-1}, \gamma y \gamma^{-1})\sigma(\gamma x y \gamma^{-1}) &= \\ \frac{\zeta(\gamma, x)\zeta(\gamma x, \gamma^{-1})\zeta(\gamma, y)\zeta(\gamma y, \gamma^{-1})}{\zeta(\gamma, xy)\zeta(\gamma xy, \gamma^{-1})} \frac{\zeta^\gamma(x, y)}{\zeta(x, y)} \zeta(x, y)\sigma^\gamma(xy) &= \\ \zeta(\gamma^{-1}, \gamma) \frac{\nu_\gamma(x)\nu_\gamma(y)}{\nu_\gamma(xy)} \frac{\zeta^\gamma(x, y)}{\zeta(x, y)} \zeta(x, y)\sigma^\gamma(xy) &= \\ \zeta(\gamma^{-1}, \gamma)\zeta(x, y)\sigma^\gamma(xy), \end{aligned}$$

where we have used (i) in the last equality.  $\square$

Equation (9) makes it convenient to restrict the discussion to normalised multipliers.

**Definition.** A multiplier  $\zeta$  on a group  $\Gamma$  is normalised if  $\zeta(x, x^{-1}) = 1$  for all  $x \in \Gamma$ . A projective representation  $\pi$  of a group  $\Gamma$  is normalised if  $\pi(x^{-1}) = \pi(x)^{-1}$  for all  $x \in \Gamma$ .

(Some authors, see e.g. Page 142 of [BeZh–98], use “normalised” for multipliers in a different meaning.)

**Lemma 7.** (i) Any multiplier  $\zeta'$  on a group is cohomologous to a normalised multiplier  $\zeta$ .

(ii) If  $\zeta$  is a normalised multiplier on a group  $\Gamma$ , then

$$(10) \quad \zeta(y^{-1}, x^{-1}) = \zeta(x, y)$$

for all  $x, y \in \Gamma$ .

*Proof.* (i) Let  $\pi'$  be an arbitrary  $\zeta'$ -representation of  $\Gamma$  on a Hilbert space  $\mathcal{H}$ . Define  $J = \{\gamma \in \Gamma \mid \gamma^2 = e\}$  and choose a partition  $\Gamma = J \sqcup K \sqcup L$  such that  $\ell \in L$  if and only if  $\ell^{-1} \in K$ . For each  $\gamma \in J$ , choose  $z_\gamma \in \mathbf{T}$  such that  $(z_\gamma \pi'(\gamma))^2 = \text{id}_{\mathcal{H}}$ . Define a projective representation  $\pi$  of  $\Gamma$  on  $\mathcal{H}$  by  $\pi(\gamma) = z_\gamma \pi'(\gamma)$  if  $\gamma \in J$ , by  $\pi(\gamma) = \pi'(\gamma)$  if  $\gamma \in K$ , and by  $\pi(\gamma) = \pi'(\gamma^{-1})^{-1}$  if  $\gamma \in L$ . Then  $\pi$  is a normalised projective representation of which the multiplier  $\zeta$  is normalised and cohomologous to  $\zeta'$ .

(ii) This is a consequence of the identity

$$\pi(x^{-1})\pi(y^{-1}) = \frac{1}{\zeta(x, y)}\pi(y^{-1}x^{-1}),$$

which is a way of writing Equation (3) when  $\pi$  is normalised.  $\square$

**Proposition 8.** Let  $\Gamma$  be a group,  $\zeta \in Z^2(\Gamma, \mathbf{T})$  a normalised multiplier, and  $N$  a normal subgroup of  $\Gamma$ . Let  $\sigma$  be a  $\zeta$ -representation of  $N$ ; for  $\gamma \in \Gamma$ , define  $\sigma^\gamma$  as in Lemma 6.

(i) The mapping

$$\sigma^\gamma : N \longrightarrow \mathcal{U}(\mathcal{H}_\sigma)$$

is a  $\zeta$ -representation of  $N$ .

(ii) We have

$$(11) \quad \sigma^{\gamma_1 \gamma_2} = (\sigma^{\gamma_1})^{\gamma_2}$$

for all  $\gamma_1, \gamma_2 \in \Gamma$ .

*Proof.* Claim (i) follows from Lemma 6, because  $\zeta$  is normalised, and checking Claim (ii) is straightforward.  $\square$

**Second standing assumption.** All multipliers appearing from now on in this paper are assumed to be normalised.

It is convenient to define now projective analogues of  $\Gamma$ -kernels, and  $\Gamma$ -faithfulness, a notion already used in the formulation of Theorem 4.

**Definitions.** Let  $\Gamma$  be a group,  $N$  a normal subgroup,  $\zeta \in Z^2(\Gamma, \mathbf{T})$  a multiplier, and  $\sigma : N \longrightarrow \mathcal{U}(\mathcal{H})$  a  $\zeta$ -representation of  $N$ .

(i) The projective  $\Gamma$ -kernel of  $\sigma$  is the normal subgroup

$$\begin{aligned} \text{Pker}_\Gamma(\sigma) &= \{x \in \text{Pker}(\sigma) \mid \sigma^\gamma(x) = \sigma(x) \text{ for all } \gamma \in \Gamma\} \\ (12) \quad &= \text{Pker} \left( \bigoplus_{\gamma \in \Gamma} \sigma^\gamma \right). \end{aligned}$$

of  $\Gamma$ .

(ii) The projective representation  $\sigma$  is  $\Gamma$ -P-faithful if  $\text{Pker}_\Gamma(\sigma) = \{e\}$ .

**Remarks.** (A) In the particular case  $\zeta = 1$ , observe that  $\text{Pker}_\Gamma(\sigma)$  is a subgroup of  $\ker_\Gamma(\sigma)$  which can be a proper subgroup.

(B) Suppose that  $N$  is a central subgroup in  $\Gamma$ . For a  $\zeta$ -representation  $\sigma$  of  $N$ , we have

$$\text{Pker}_\Gamma(\sigma) = \{x \in \text{Pker}(\sigma) \mid \zeta(\gamma x, \gamma^{-1})\zeta(\gamma, x) = 1 \text{ for all } \gamma \in \Gamma\}.$$

Since

$$\zeta(\gamma x, \gamma^{-1})\zeta(x, \gamma) = \zeta(x, \gamma)\zeta(x\gamma, \gamma^{-1}) = \zeta(x, 1)\zeta(\gamma, \gamma^{-1}) = 1$$

for every  $x \in Z(\Gamma)$  and  $\gamma \in \Gamma$  (recall that  $\zeta$  is normalised), we have  $\zeta(\gamma x, \gamma^{-1}) = \zeta(x, \gamma)^{-1}$  and therefore also

$$(13) \quad \text{Pker}_\Gamma(\sigma) = \{x \in \text{Pker}(\sigma) \mid \zeta(x, \gamma) = \zeta(\gamma, x) \text{ for all } \gamma \in \Gamma\}.$$

#### 4. Extensions of groups associated to multipliers

##### Proof of Theorem 1

Consider a group  $\Gamma$ , a multiplier  $\zeta \in Z^2(\Gamma, \mathbf{T})$ , and a subgroup  $A$  of  $\mathbf{T}$  containing  $\zeta(\Gamma \times \Gamma)$ .

We define a group  $\Gamma(\zeta)$  with underlying set  $A \times \Gamma$  and multiplication

$$(14) \quad (s, x)(t, y) = (st\zeta(x, y), xy)$$

for all  $s, t \in A$  and  $x, y \in \Gamma$ ; observe that  $(s, x)^{-1} = (s^{-1}, x^{-1})$ , because  $\zeta$  is normalised. This fits naturally in a central extension

$$(15) \quad \{e\} \longrightarrow A \xrightarrow{s \mapsto (s, e)} \Gamma(\zeta) \xrightarrow{(s, x) \mapsto x} \Gamma \longrightarrow \{e\}.$$

We insist on the fact that  $\Gamma(\zeta)$  depends on the choice of  $A$ , even if the notation does not show it. Whenever  $H$  is a subgroup of  $\Gamma$ , we identify  $H(\zeta)$  with the appropriate subgroup of  $\Gamma(\zeta)$ .

To any  $\zeta$ -representation  $\pi$  of  $\Gamma$  on some Hilbert space  $\mathcal{H}$  corresponds a representation  $\pi^0$  of  $\Gamma(\zeta)$  on the same space defined by

$$(16) \quad \pi^0(s, x) = s\pi(x)$$

for all  $(s, x) \in \Gamma(\zeta)$ . Conversely, to any representation  $\pi^0$  of  $\Gamma(\zeta)$  on  $\mathcal{H}$  which is the identity on  $A$  (namely which is such that  $\pi^0(a) = a \text{id}_{\mathcal{H}}$  for all  $a \in A$ ) corresponds a  $\zeta$ -representation  $\pi$  of  $\Gamma$  defined by

$$(17) \quad \pi(x) = \pi^0(1, x).$$

**Lemma 9.** (i) *The correspondance  $\pi \longleftrightarrow \pi^0$  given by Equations (16) and (17) is a bijection between  $\zeta$ -representations of  $\Gamma$  on  $\mathcal{H}$  and representations of  $\Gamma(\zeta)$  on  $\mathcal{H}$  which are the identity on  $A$ .*

(ii) *If  $\pi$  and  $\pi^0$  correspond to each other in this way,  $\pi$  is irreducible if and only if  $\pi^0$  is so.*

*Assume moreover that the subgroup  $A$  of  $\mathbf{T}$  contains both the image  $\zeta(\Gamma \times \Gamma)$  of the multiplier and the subset*

$$(18) \quad T_\pi := \{z \in \mathbf{T} \mid z \text{id}_{\mathcal{H}} = \pi(x) \text{ for some } x \in \text{Pker}(\pi)\}$$

*of  $\mathbf{T}$ .*

(iii) *If  $\pi$  and  $\pi^0$  are as above,  $\pi$  is P-faithful if and only if  $\pi^0$  is faithful.*

*Observation.* If  $\Gamma$  is countable,  $\zeta(\Gamma \times \Gamma)$  and  $T_\pi$  are countable subsets of  $\mathbf{T}$ , so that there exists a countable group  $A$  as in (15) which contains both  $T_\pi$  and the image of  $\zeta$ .

*Proof.* Claims (i) and (ii) are obvious. The generalisation of Claim (i) for continuous representations of locally compact groups appears as a corollary to Theorem 1 in [Klep–74]; see also Theorem 2.1 in [Mack–58].

For Claim (iii), suppose first that  $\pi$  is P-faithful. If  $(s, x) \in \ker(\pi^0)$ , namely if  $s\pi(x) = 1$ , then  $x \in \text{Pker}(\pi)$ , so that  $x = e$ ; it follows that  $s = 1$ , so that  $(s, x) = (1, e)$ . Thus  $\pi^0$  is faithful.

Suppose now that  $\pi^0$  is faithful. If  $x \in \text{Pker}(\pi)$ , namely if there exists  $s \in \mathbf{T}$  such that  $s\pi(x) = 1$ , then  $(s, x) \in \ker(\pi^0)$ , so that  $s = 1$  and  $x = e$ . Thus  $\pi$  is P-faithful.  $\square$

**Proof of Theorem 1.** Let  $\pi$  be an irreducible P-faithful projective representation of  $\Gamma$ , of multiplier  $\zeta$ . Choose a subgroup  $A$  of  $\mathbf{T}$  containing  $\zeta(\Gamma \times \Gamma)$  and  $T_\pi$  (as defined in Lemma 9). Let  $\Gamma(\zeta)$  be as in (14) and  $\pi^0$  be as in (16). Since  $\pi^0$  is irreducible and faithful (Lemma 9), Schur's Lemma implies that  $A$  is the centre of  $\Gamma(\zeta)$ , so that  $\Gamma \approx \Gamma(\zeta)/Z(\Gamma(\zeta))$ .

If  $\Gamma$  is countable,  $\Gamma(\zeta)$  can be chosen countable, by the observation just after Lemma 9.

Conversely, let  $\Delta$  be a group such that  $\Gamma \approx \Delta/Z(\Delta)$  and let  $\pi^0$  be a representation of  $\Delta$  which is irreducible and faithful. Again by Schur's Lemma, the subgroup  $(\pi^0)^{-1}(\mathbf{T})$  coincides with the centre of  $\Delta$ . Let  $\mu : \Gamma \rightarrow \Delta$  be any set-theoretical section of the projection  $\Delta \rightarrow \Delta/Z(\Delta) \approx \Gamma$ , with  $\mu(e_\Gamma) = e_\Delta$ . The assignment  $\pi : \gamma \mapsto \pi^0(\mu(\gamma))$  defines a projective representation of  $\Gamma$  which is irreducible and P-faithful, by Lemma 9.  $\square$

Our next lemma reduces essentially to Lemma 9.iii if  $N = \Gamma$ . It will be used in the proof of Lemma 13.

**Lemma 10.** *Consider a normal subgroup  $N$  of  $\Gamma$  and a  $\zeta$ -representation  $\sigma$  of  $N$  in some Hilbert space  $\mathcal{H}$ . Let  $A$  be a subgroup of  $\mathbf{T}$  containing both  $\zeta(N \times N)$  and the subset*

$$(19) \quad T_{\sigma, \Gamma} := \{z \in \mathbf{T} \mid z \text{ id}_{\mathcal{H}} = \sigma(x) \text{ for some } x \in \text{Pker}_{\Gamma}(\sigma)\}$$

*of  $\mathbf{T}$  (compare with Equation (18)). Define  $N(\zeta)$  and  $\Gamma(\zeta)$  as in the beginning of the present section. Then*

- (i)  $(\sigma^\gamma)^0 = (\sigma^0)^\gamma$  for all  $\gamma \in \Gamma$  ;
- (ii)  $\text{Pker}_{\Gamma}(\sigma) = \left\{ x \in N \mid \begin{array}{l} \text{there exists } s \in A \\ \text{with } (s, x) \in \ker_{\Gamma(\zeta)}(\sigma^0) \end{array} \right\}$ , so that, in particular,  $\sigma$  is  $\Gamma$ -P-faithful if and only if  $\sigma^0$  is  $\Gamma(\zeta)$ -faithful.

*Proof.* Checking (i) is straightforward.

To show (ii), let  $x \in \text{Pker}_{\Gamma}(\sigma)$ . Thus there exists  $s \in A$  such that  $\sigma^\gamma(x) = s^{-1} \text{id}_{\mathcal{H}}$  for all  $\gamma \in \Gamma$ . Then, for all  $\gamma \in \Gamma$ , we have

$$(\sigma^0)^\gamma(s, x) = (\sigma^\gamma)^0(s, x) = s\sigma^\gamma(x) = ss^{-1} \text{id}_{\mathcal{H}} = \text{id}_{\mathcal{H}},$$

that is,  $(s, x) \in \ker_{\Gamma(\zeta)}(\sigma^0)$ .

Conversely, let  $x \in N$  be such that there exists  $s \in A$  with  $(s, x) \in \ker_{\Gamma(\zeta)}(\sigma^0)$ . Then, for all  $\gamma \in \Gamma$ , we have

$$\sigma^\gamma(x) = (\sigma^\gamma)^0(1, x) = s^{-1}(\sigma^\gamma)^0(s, x) = s^{-1}(\sigma^0)^\gamma(s, x) = s^{-1} \text{id}_{\mathcal{H}},$$

that is,  $x \in \text{Pker}_{\Gamma}(\sigma)$ .  $\square$

Given a group  $\Gamma$  and a multiplier  $\zeta \in Z^2(\Gamma, \mathbf{T})$ , a  $\zeta$ -character of  $\Gamma$  is a  $\zeta$ -representation  $\chi : \Gamma \rightarrow \mathbf{T} = \mathcal{U}(\mathbf{C})$ ; we denote by  $X^\zeta(\Gamma)$  the set of all these. Observe that, for  $\chi_1, \chi_2 \in X^\zeta(\Gamma)$ , the product  $\chi_1 \overline{\chi_2}$  is a character of  $\Gamma$  in the usual sense, namely a homomorphism from  $\Gamma$  to  $\mathbf{T}$ . Such a homomorphism factors via the *abelianisation*  $\Gamma/[\Gamma, \Gamma]$ , that we denote by  $\Gamma^{\text{ab}}$ . We denote by  $\widehat{\Gamma^{\text{ab}}}$  the *character group*  $\text{Hom}(\Gamma, \mathbf{T})$ . For further reference, we state here the following straightforward observations.

**Lemma 11.** *Let  $\Gamma$  be a group and let  $\zeta \in Z^2(\Gamma, \mathbf{T})$ .*

(i) *If  $\zeta \notin B^2(\Gamma, \mathbf{T})$ , then  $X^\zeta(\Gamma) = \emptyset$ .*

(ii) *If  $\zeta \in B^2(\Gamma, \mathbf{T})$ , there exists a bijection between  $X^\zeta(\Gamma)$  and  $\widehat{\Gamma^{\text{ab}}}$ .*

*Proof.* (i) Suppose that there exists  $\chi \in X^\zeta(\Gamma)$ ; then  $\zeta(x, y) = \frac{\chi(x)\chi(y)}{\chi(xy)}$  for all  $x, y \in \Gamma$ , so that  $\zeta$  is a coboundary.

(ii) If  $\zeta \in B^2(\Gamma, \mathbf{T})$ , there exists a mapping  $\nu : \Gamma \rightarrow \mathbf{T}$  such that  $\zeta$  is related to  $\nu$  as in Formula (5), so that  $\nu \in X^\zeta(\Gamma)$ . For any  $\chi \in X^\zeta(\Gamma)$ , observe that  $\chi\nu$  is an ordinary character of  $\Gamma$ , so that  $\chi \mapsto \chi\nu$  is a bijection  $X^\zeta(\Gamma) \rightarrow X^1(\Gamma) = \widehat{\Gamma^{\text{ab}}}$ .  $\square$

### 5. Proof of (i) $\implies$ (ii) $\iff$ (iii) in Theorem 4

The first proposition of this section is a reminder of Section 3 of [Mack–58].

**Proposition 12** (Mackey). *A  $\zeta$ -representation  $\pi$  of a countable group  $\Gamma$  has a direct integral decomposition in irreducible  $\zeta$ -representations, of the form*

$$(20) \quad \pi = \int_{\Omega}^{\oplus} \pi_{\omega} d\mu(\omega).$$

*Proof.* Consider a subgroup  $A$  of  $\mathbf{T}$  containing  $\zeta(\Gamma \times \Gamma)$  and the subset  $T_{\pi}$  defined in (18), the resulting extension  $\Gamma(\zeta)$ , and the representation  $\pi^0$  of  $\Gamma(\zeta)$  defined in (16). There exists a direct integral decomposition in irreducible representations

$$\pi^0 = \int_{\Omega}^{\oplus} (\pi^0)_{\omega} d\mu(\omega)$$

with respect to a measurable field  $\omega \mapsto (\pi^0)_{\omega}$  of irreducible representations of  $\Gamma(\zeta)$  on a measure space  $(\Omega, \mu)$ ; see [Di–69C\*], Sections 8.5 and 18.7.

Since  $\pi^0(s, x) = s\pi(x)$  for all  $(s, x) \in \Gamma(\zeta)$ , we have  $(\pi^0)_{\omega}(s, x) = s(\pi^0)_{\omega}(1, x)$  for all  $(s, x) \in \Gamma(\zeta)$  and for almost all  $\omega \in \Omega$ . It follows that, for almost all  $\omega \in \Omega$ , the mapping  $\pi_{\omega} : \Gamma \rightarrow \mathcal{U}(\mathcal{H}_{\omega})$  defined by  $\pi_{\omega}(x) = (\pi^0)_{\omega}(1, x)$  is a  $\zeta$ -representation of  $\Gamma$  which is irreducible, and  $(\pi_{\omega})^0 = (\pi^0)_{\omega}$ . Hence we have a decomposition as in (20).  $\square$

We isolate in the next lemma an argument that we will use in the proofs of Propositions 14, 15, and 16.

**Notation.** Let  $\Gamma$  be a group and  $N$  a normal subgroup. We denote by  $(C_j)_{j \in J}$  the  $\Gamma$ -conjugacy classes contained in  $N$  and, for each  $j \in J$ , by  $N_j$  the normal subgroup of  $\Gamma$  generated by  $C_j$ .

**Lemma 13.** *Let  $\Gamma$  be a group,  $\zeta \in Z^2(\Gamma, \mathbf{T})$ , and  $N$  a normal subgroup of  $\Gamma$ . Let  $(C_j)_{j \in J}$  and  $(N_j)_{j \in J}$  be as above. Let  $A$  be a subgroup of  $\mathbf{T}$  containing  $\zeta(\Gamma \times \Gamma)$ , as well as  $\chi(x)$  for every  $\chi \in X^\zeta(N_j)$ ,  $j \in J$ , and  $x \in N_j$ . Let  $N(\zeta)$  be the central extension of  $N$  corresponding to  $\zeta$  and  $A$  as in (15).*

*Then, for every  $\zeta$ -representation  $\sigma$  of  $N$ , we have:  $\sigma$  is  $\Gamma$ - $P$ -faithful if and only if the corresponding representation  $\sigma^0$  of  $N(\zeta)$  is  $\Gamma(\zeta)$ -faithful.*

*Remark.* This lemma will be applied in situations where  $\Gamma$  is a countable group. Observe that, if  $\Gamma$  is countable, there exists a countable group  $A$  as in the previous lemma as soon as  $\Gamma$  has Property (Fab), or more generally as soon as  $N_j^{\text{ab}}$  is finite for all  $j \in J$  such that the restriction of  $\zeta$  to  $N_j$  is in  $B^2(N_j, \mathbf{T})$ .

*Proof.* In view of Lemma 10, it suffices to prove that  $A$  contains  $T_{\sigma, \Gamma}$  for every  $\zeta$ -representation  $\sigma$  of  $N$ .

Let  $z \in T_{\sigma, \Gamma}$ ; choose  $x \in \text{Pker}_\Gamma(\sigma)$  such that  $\sigma(x) = \text{id}_{\mathcal{H}_\sigma}$ . Let  $j \in J$  be such that  $x \in C_j$ ; we have  $N_j \subset \text{Pker}_\Gamma(\sigma)$ , because the latter group is normal in  $\Gamma$ . The restriction of  $\sigma$  to  $N_j$  defines a  $\zeta$ -character  $\chi \in X^\zeta(N_j)$  such that  $z = \chi(x)$ . This shows that  $z \in A$ , by the choice of  $A$ .  $\square$

Implications (i)  $\implies$  (ii) and (i)  $\implies$  (iii) of Theorem 4 are particular cases of the following proposition, because the minisocle  $MS(\Gamma)$  and the subgroup  $MA(\Gamma)$  of a countable group  $\Gamma$  have the properties assumed for the group  $N$  below (Proposition 1 in [BeHa–08]).

**Proposition 14.** *Let  $\Gamma$  be a countable group,  $N$  a normal subgroup, and  $\zeta \in Z^2(\Gamma, \mathbf{T})$ . Let  $(C_j)_{j \in J}$  and  $(N_j)_{j \in J}$  be as just before Lemma 13. Assume that the abelianised group  $N_j^{\text{ab}}$  is finite for all  $j \in J$  such that the restriction to  $N_j$  of  $\zeta$  is in  $B^2(N_j, \mathbf{T})$ .*

*Let  $\pi$  be a  $\zeta$ -representation of  $\Gamma$  and let*

$$(21) \quad \sigma := \pi|_N = \int_{\Omega}^{\oplus} \sigma_{\omega} d\mu(\omega)$$

*be a direct integral decomposition of the restriction of  $\pi$  to  $N$  in irreducible  $\zeta$ -representations  $\sigma_{\omega}$  of  $N$ .*

*If  $\pi$  is irreducible and  $P$ -faithful, then  $\sigma_{\omega}$  is  $\Gamma$ - $P$ -faithful for almost all  $\omega \in \Omega$ .*

*Proof.* The strategy is to reduce the proof to the case of ordinary representations and to use Lemma 9 of [BeHa–08].

By hypothesis and by Lemma 11,  $X^\zeta(N_j)$  is finite (possibly empty) for all  $j \in J$ . Since  $J$  is countable, we can choose a *countable* subgroup



$A$  of  $\mathbf{T}$  containing  $\zeta(\Gamma \times \Gamma)$ , as well as  $\chi(x)$  for every  $\chi \in X^\zeta(N_j)$ ,  $j \in J$ , and  $x \in N_j$ .

Let  $\Gamma(\zeta)$  and  $N(\zeta)$  be as in (14); let  $\pi^0$  and  $\sigma_\omega^0$  be the representations of  $\Gamma(\zeta)$  and  $N(\zeta)$  corresponding to the  $\zeta$ -representations  $\pi$  and  $\sigma_\omega$ , respectively. Because  $\pi$  is P-faithful, the subset  $T_\pi$  defined in (18) is reduced to  $\{e\}$  and therefore  $\pi^0$  is faithful (Lemma 9).

Since (see the proof of Proposition 12)

$$\sigma^0 = \pi^0|_N = \int_{\Omega}^{\oplus} \sigma_{\omega}^0 d\mu(\omega),$$

the representation  $\sigma_{\omega}^0$  of  $N(\zeta)$  is  $\Gamma(\zeta)$ -faithful for almost all  $\omega$  (Lemma 9 of [BeHa–08]). Therefore, by Lemma 13,  $\sigma_{\omega}$  is  $\Gamma$ -P-faithful for almost all  $\omega$ .  $\square$

The equivalence (ii)  $\iff$  (iii) of Theorem 4 is a particular case of the following Proposition.

**Proposition 15.** *Assume that the normal subgroup  $N$  of  $\Gamma$  is a direct product  $B \times S$  of normal subgroups of  $\Gamma$ , and that  $S = \prod_{i \in I} S_i$  is a restricted direct product of finite simple nonabelian subgroups  $S_i$ . Assume moreover that any  $\Gamma$ -invariant subgroup of  $B$  generated by one  $\Gamma$ -conjugacy class has finite abelianisation.*

*The following conditions are equivalent:*

- ( $\alpha$ )  $N$  has a  $\Gamma$ -P-faithful irreducible  $\zeta$ -representation;
- ( $\beta$ )  $B$  has a  $\Gamma$ -P-faithful irreducible  $\zeta$ -representation.

*Proof* The proof of the implication ( $\alpha$ )  $\Rightarrow$  ( $\beta$ ) follows closely the proof of Proposition 14, with one difference: one has to use the more general version of Lemma 9 in [BeHa–08] which is mentioned at the bottom of page 866 of this article.

For the converse implication, we assume now that  $B$  has a  $\Gamma$ -P-faithful irreducible  $\zeta$ -representation  $\sigma$ . The group  $S$  has a faithful irreducible (unitary) representation, say  $\rho$ , such that  $\rho(x) \notin \mathbf{T}$  for all  $x \in S, x \neq e$ , namely  $\rho$  is P-faithful; see the proof of Lemma 13 in [BeHa–08] (this Lemma 13 contains a hypothesis "A abelian", but it is redundant for the part of the proof we need here). The tensor product  $\sigma \otimes \rho$  is an irreducible  $\zeta$ -representation of  $N$ . Since  $\sigma$  is  $\Gamma$ -P-faithful, it follows from Lemma 12 of [BeHa–08] that  $\sigma \otimes \rho$  is  $\Gamma$ -P-faithful.  $\square$

## 6. End of proof of Theorem 4

Let us first recall the definition of induction for projective representations, from Section 4 in [Mack–58].

Let  $\Gamma$  be a group,  $\zeta \in Z^2(\Gamma, \mathbf{T})$  a multiplier,  $H$  a subgroup of  $\Gamma$ , and  $\sigma : H \rightarrow \mathcal{U}(\mathcal{K})$  a  $\zeta$ -representation. Let  $\mathcal{H}$  be the Hilbert space of mappings  $f : \Gamma \rightarrow \mathcal{K}$  with<sup>6</sup> the two following properties:

- $f(hx) = \zeta(h, x) \sigma(h)(f(x))$  for all  $x \in \Gamma$  and  $h \in H$ ,
- $\sum_{x \in \Gamma \setminus H} \|f(x)\|^2 < \infty$ .

The  $\zeta$ -representation  $\text{Ind}_H^\Gamma(\sigma)$  of  $\Gamma$  is the multiplier representation of  $\Gamma$  in  $\mathcal{H}$  defined by

$$(22) \quad (\text{Ind}_H^\Gamma(\sigma)(x)f)(y) = f(yx)$$

for all  $x, y \in \Gamma$ .

It can be checked (see [Mack–58], Pages 273-4) that the representation  $(\text{Ind}_H^\Gamma(\sigma))^0$  of  $\Gamma(\zeta)$  associated to  $\text{Ind}_H^\Gamma(\sigma)$  is the representation  $\text{Ind}_{H(\zeta)}^{\Gamma(\zeta)}(\sigma^0)$  induced by the representation  $\sigma^0$  from  $H(\zeta)$  to  $\Gamma(\zeta)$ .

The last claim of Theorem 4 follows from the next proposition.

**Proposition 16.** *Let  $\Gamma$  be a countable group and let  $\zeta \in Z^2(\Gamma, \mathbf{T})$ . Let  $(C_j)_{j \in J}$  and  $(N_j)_{j \in J}$  be as just before Lemma 13, with  $N = \Gamma$ . Assume that the abelianised group  $N_j^{\text{ab}}$  is finite for all  $j \in J$  such that the restriction to  $N_j$  of  $\zeta$  is in  $B^2(N_j, \mathbf{T})$ .<sup>7</sup>*

*Let  $\sigma$  be a  $\zeta$ -representation of the minisocle  $MS(\Gamma)$ . Set  $\pi := \text{ind}_{MS(\Gamma)}^\Gamma(\sigma)$  and let*

$$(23) \quad \pi = \int_{\Omega}^{\oplus} \pi_{\omega} d\mu(\omega)$$

*be a direct integral decomposition of  $\pi$  in irreducible  $\zeta$ -representations of  $\Gamma$ .*

*If  $\sigma$  is irreducible and  $\Gamma$ - $P$ -faithful, then  $\pi_{\omega}$  is  $P$ -faithful for almost all  $\omega \in \Omega$ .*

*Proof.* As for Proposition 14, the strategy is to reduce the proof to the case of ordinary representations, and to use this time Lemma 10 of [BeHa–08]. We write  $M$  for  $MS(\Gamma)$ .

By hypothesis and by Lemma 11, we can choose a countable subgroup  $A$  be a  $\mathbf{T}$  containing the sets  $\zeta(\Gamma \times \Gamma)$  and  $X^\zeta(N_j)(x)$  for every  $j \in J$  and every  $x \in N_j$ . We consider the corresponding extension  $\Gamma(\zeta)$  of  $\Gamma$ . Denote by  $\pi^0$  and  $\pi_{\omega}^0$  the representations of  $\Gamma(\zeta)$  corresponding to the  $\zeta$ -representations  $\pi$  and  $\pi_{\omega}$  of  $\Gamma$ , and similarly  $\sigma^0$  for the representation of  $M(\zeta)$  corresponding to the  $\zeta$ -representation  $\sigma$  of  $M$ .

<sup>6</sup>We use  $H \setminus \Gamma$ , rather than  $\Gamma/H$  as in [BeHa–08], which provides easier formulas.

<sup>7</sup>This assumption holds whenever  $\Gamma$  has Property (Fab).

We have

$$\pi^0 = \text{ind}_{M(\zeta)}^{\Gamma(\zeta)} \sigma^0 = \int_{\Omega}^{\oplus} \pi_{\omega}^0 d\mu(\omega).$$

In view of Lemma 13 applied to  $N = \Gamma$ , it suffices to show that  $\pi_{\omega}^0$  is P-faithful for almost all  $\omega$ .

Since  $\sigma$  is  $\Gamma$ -P-faithful, we have that  $\sigma^0$  is  $\Gamma$ -faithful (again by Lemma 13). It will follow from Lemma 10 in [BeHa-08] that  $\pi_{\omega}^0$  is faithful for almost all  $\omega$  provided we show that  $M(\zeta) \cap L \neq \{e\}$ , for every finite foot  $L$  in  $\Gamma(\zeta)$ .

In order to check this condition, let  $L$  be finite foot in  $\Gamma(\zeta)$ . We claim that  $L \subset M(\zeta)$ . Indeed, recall that, set-theoretically, we have  $\Gamma(\zeta) = A \times \Gamma$  and  $M(\zeta) = A \times M$ ; thus, for any  $(t, y) \in L$  with  $(t, y) \neq e$ , we have  $y \in M (= MS(\Gamma))$ , and therefore  $(t, y) \in M(\zeta)$ .  $\square$

## 7. Capable and incapable groups

### Proof of Proposition 2

In Proposition 2, Claims (i) and (ii) are respectively Corollary 2.3 and part of Corollary 2.2 of [BeFS-79]. Claim (iii) is a consequence of Claim (i), in a formulation and with a proof shown to us by Graham Ellis [Ellis], see below. Corollary 3 is an immediate consequence of Theorem 1 and Proposition 2.

**Proof of Claim (iii) in Proposition 2.** For a central extension

$$\{e\} \rightarrow A \rightarrow \Gamma \rightarrow \Gamma/A \rightarrow \{e\},$$

the Ganea extension of the Hochschild-Serre exact sequence in homology with trivial coefficients  $\mathbf{Z}$  is

$$\begin{array}{ccccccc} A \otimes_{\mathbf{Z}} \Gamma^{\text{ab}} & \longrightarrow & H_2(\Gamma, \mathbf{Z}) & \longrightarrow & H_2(\Gamma/A, \mathbf{Z}) & \longrightarrow & \\ A & \longrightarrow & H_1(\Gamma, \mathbf{Z}) & \longrightarrow & H_1(\Gamma/A, \mathbf{Z}) & \longrightarrow & \{0\} \end{array}$$

(see for example [EcHS-72]). If  $\Gamma$  is perfect (so that  $\Gamma^{\text{ab}} = \{0\}$ ), this reduces to

$$\{0\} \longrightarrow H_2(\Gamma, \mathbf{Z}) \longrightarrow H_2(\Gamma/A, \mathbf{Z}) \longrightarrow A \longrightarrow \{0\}$$

and it follows from the definition of the epicentre of  $\Gamma$  that  $Z^*(\Gamma) = Z(\Gamma)$ . Thus Claim (iii) is a straightforward consequence of Claim (i) of Proposition 2.  $\square$

It is well-known that any cyclic group  $C \neq \{e\}$  is incapable. Indeed, suppose *ab absurdo* that  $C = \Delta/Z(\Delta)$ . Choose a generator  $s$  of  $C$  and a preimage  $t$  of  $s$  in  $\Delta$ ; any  $\delta \in \Delta$  can be written as  $\delta = zt^j$  for some  $z \in Z(\Delta)$  and  $j \in \mathbf{Z}$ , and two elements of this kind commute with each other, so that  $\Delta$  is abelian, hence  $Z(\Delta) = \Delta$ , incompatible with

$C \neq \{e\}$ . The next lemma, which appears on Page 137 of [Hall–40], rests on an elaboration of the same argument.

**Lemma 17.** *Let  $\Gamma$  be a group containing an element  $s_0 \neq e$  such that the set*

$$\{s \in \Gamma \mid \text{there exists } n \in \mathbf{Z} \text{ with } s^n = s_0\}$$

*(where  $n$  can depend on  $s$ ) generates  $\Gamma$ . Then  $\Gamma$  is incapable.*

*Proof.* It suffices to show that, given any central extension

$$\{e\} \longrightarrow A \longrightarrow \Delta \xrightarrow{\pi} \Gamma \longrightarrow \{e\},$$

$s_0$  has a preimage  $t_0$  in  $\Delta$  which is central.

Let  $\delta \in \Delta$ . There exists

$$s_1, \dots, s_k \in \Gamma \quad \text{and} \quad j_1, \dots, j_k, n_1, \dots, n_k \in \mathbf{Z}$$

such that

$$\pi(\delta) = s_1^{j_1} \cdots s_k^{j_k} \quad \text{and} \quad s_1^{n_1} = \cdots = s_k^{n_k} = s_0.$$

For  $i = 0, \dots, k$ , choose a preimage  $t_i$  of  $s_i$  in  $\Delta$ . There exist  $a, a_1, \dots, a_k \in A$  with

$$\delta = at_1^{j_1} \cdots t_k^{j_k} \quad \text{and} \quad a_i t_i^{n_i} = t_0 \quad \text{for } i = 1, \dots, k.$$

It follows that  $t_0$  commutes with  $t_i$  for  $i = 1, \dots, k$ , and thus that  $t_0$  commutes with  $\delta$ , as was to be shown.  $\square$

Claims (i) to (v) of the following proposition are straightforward consequences of Lemma 17, and the last claim follows from Theorem 1.

**Proposition 18.** *The following groups are incapable:*

- (i) *cyclic groups, quasicyclic groups  $\mathbf{Z}(p^\infty)$ , and the groups  $\mathbf{Z}[1/m]$  for all integers  $m \geq 2$ ;*
- (ii) *finite abelian groups  $\mathbf{Z}/d_1\mathbf{Z} \times \cdots \times \mathbf{Z}/d_m\mathbf{Z}$ , (where  $n \geq 2$ ,  $d_1, \dots, d_m \geq 2$ ,  $d_1|d_2|\cdots|d_m$ ) with  $d_{m-1} < d_m$ ;*
- (iii) *subgroups of  $\mathbf{Q}$ ;*
- (iv)  *$\mathrm{SL}_2(\mathbf{Z}) = \langle s, t \mid s^2 = t^3 \text{ is central of order } 2 \rangle$ ;*
- (v)  *$\langle s, t \mid s^m = t^n \text{ and } (s^m)^k = 1 \rangle$  for  $m, n \geq 1$ ,  $k \geq 2$ , as well as  $\langle s, t \mid s^m = t^n \rangle$ .*

*In particular, these groups do not afford any irreducible  $P$ -faithful projective representation.*

*Remarks.* About (i): for any prime  $p$ , the quasicyclic group  $\mathbf{Z}(p^\infty)$  is the subgroup of  $\mathbf{T}$  of roots of 1 of order some power of  $p$ ; equivalently,  $\mathbf{Z}(p^\infty)$  is the quotient  $\mathbf{Q}_p/\mathbf{Z}_p$  of the  $p$ -adic numbers by the  $p$ -adic integers.

About (iii): there is a classification of the subgroups of  $\mathbf{Q}$ , which is standard; see for example Chapter 10 of [Rotm–95].

About (iv), let us recall that  $\mathrm{SL}_2(\mathbf{Z})$  is generated by a square root  $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  and a cubic root  $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$  of the central matrix  $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ . Next, since  $\mathrm{SL}_2(\mathbf{Z}) = \langle s, t \mid s^2 = t^3, s^4 = 1 \rangle$  has deficiency  $\geq 0$  and finite abelianisation (indeed  $\mathrm{SL}_2(\mathbf{Z})^{\mathrm{ab}} \approx \mathbf{Z}/12\mathbf{Z}$ ), it follows from Philip Hall's Inequality<sup>8</sup> that  $H_2(\mathrm{SL}_2(\mathbf{Z}), \mathbf{Z}) = \{0\}$ .

Similarly,  $H_2(\mathrm{PSL}_2(\mathbf{Z}), \mathbf{Z}) = \{0\}$ . This follows alternatively from the formula  $H_n(\Gamma_1 * \Gamma_2, \mathbf{Z}) \approx H_n(\Gamma_1, \mathbf{Z}) \oplus H_n(\Gamma_2, \mathbf{Z})$  for  $n \geq 1$ , see Corollary 6.2.10 in [Weib–94]. But  $\mathrm{PSL}_2(\mathbf{Z})$  is capable, since its centre is trivial.

About (v): the group  $\mathrm{SL}_2(\mathbf{Z})$  is of course a particular case of groups in (v); if  $m$  and  $n$  are coprime and at least 2, the group  $\langle s, t \mid s^m = t^n \rangle$  is a *torus knot group*.

## 8. On abelian groups

The next proposition rests on a construction which appears in many places, including [Mack–49] and [Weil–64]. It is part of the *Stone–von Neumann–Mackey Theorem*, see the beginning of [MuNN–91].

Let  $L$  be an abelian group, written multiplicatively. Consider the group  $X(L) = \mathrm{Hom}(L, \mathbf{T})$  of characters of  $L$ , with the topology of the simple convergence, which makes it a locally compact abelian group. By Pontryagin duality, we can (and do) identify  $L$  to the group of continuous characters on  $X(L)$ . Consider also a dense subgroup  $M$  of  $X(L)$  and the direct product group  $L \times M$ . The mapping

$$\zeta : (L \times M) \times (L \times M) \longrightarrow \mathbf{T}, \quad ((\ell, m), (\ell', m')) \longmapsto m'(\ell)$$

is a multiplier on  $L \times M$ . Let  $A$  be a subgroup of  $\mathbf{T}$  containing the image of  $\zeta$ . By definition, the corresponding *generalised Heisenberg group* is

$$H_{L,M}^A = A \times L \times M$$

---

<sup>8</sup>Namely: for a finitely presented group  $\Gamma$ , the deficiency of  $\Gamma$  is bounded by the difference  $\dim_{\mathbf{Q}}((\Gamma)^{\mathrm{ab}} \otimes_{\mathbf{Z}} \mathbf{Q}) - s(H_2(\Gamma, \mathbf{Z}))$ , where  $s(H)$  stands for the minimum number of generators of the group  $H$ ; see e.g. Lemma 1.2 in [Epst–61]. Recall also that the *deficiency* of a finite presentation of a group is the number of its generators minus the number of its relations, and the deficiency of a finitely presented group the maximum of the deficiencies of its finite presentations.

with product defined by

$$(z, \ell, m)(z', \ell', m') = (zz'\ell(m'), \ell\ell', mm').$$

It is routine to check that the centre of  $H_{L,M}^A$  is  $A$ .

**Proposition 19.** *Any abelian group of the form  $L \times M$ , with  $M$  dense in  $X(L)$  as above, affords a projective representation which is irreducible and  $P$ -faithful.*

*Proof.* Let us sketch the definition and some properties of the “Stone–von Neumann–Mackey representation” of  $H_{L,M}^A$  on  $\ell^2(L)$ ; the latter is a Hilbert space, with scalar product defined by  $\langle \xi | \eta \rangle = \sum_{\ell \in L} \overline{\xi(\ell)} \eta(\ell)$ .

For  $(z, \ell, m) \in H_{L,M}^A$  and  $\xi \in \ell^2(L)$ , set

$$(R(z, \ell, m)\xi)(x) = zm(x)\xi(x\ell) \quad \text{for all } x \in L.$$

It can be checked that  $R(z, \ell, m)$  is a unitary operator on  $\mathcal{H} = \ell^2(L)$  and that

$$R : H_{L,M}^A \longrightarrow \mathcal{U}(\mathcal{H})$$

is a representation of  $H_{L,M}^A$  on  $\mathcal{H}$ .

The space  $\mathcal{H}$  has a natural orthonormal basis  $(\delta_u)_{u \in L}$ . It is easy to check that

$$R(z, \ell, m)\delta_u = zm(u\ell^{-1})\delta_{u\ell^{-1}},$$

so that the representation  $R$  is faithful. Observe that, for all  $u \in L$  and  $m \in M$ , the vector  $\delta_u$  is an eigenvector of  $R(1, 1, m)$  with eigenvalue  $m(u)$ . If

$$V_u = \{\xi \in \mathcal{H} \mid R(1, 1, m)\xi = m(u)\xi \text{ for all } m \in M\},$$

then  $V_u = \mathbf{C}\delta_u$  is an eigenspace of dimension 1 and  $\mathcal{H} = \bigoplus_{u \in L} V_u$  (Hilbert sum).

Let now  $S \in \mathcal{L}(\mathcal{H})$  be an operator commuting with  $R(1, 1, m)$  for all  $m \in M$ . Since  $M$  is dense in  $X(L)$ , for every  $u, v \in L$  with  $u \neq v$ , there exists  $m \in M$  such that  $m(u) \neq m(v)$ . As is easily checked, this implies that  $S$  is diagonal with respect to the basis  $(\delta_u)_{u \in L}$ , namely that there exist complex numbers  $s_u$  such that  $S(\delta_u) = s_u\delta_u$  for all  $u \in M$ . Suppose moreover that  $S$  commutes with  $R(1, \ell, 1)$  for all  $\ell \in L$ ; since  $R(1, \ell, 1)\delta_u = \delta_{u\ell^{-1}}$ , we have  $s_u = s_{u\ell^{-1}}$  for all  $u, \ell \in L$ . Thus  $S$  is a scalar multiple of the identity operator. It follows from Schur’s lemma that the representation  $R$  is irreducible.

The representation  $R$  of  $H_{L,M}^A$  provides a projective representation of  $L \times M$  which is irreducible and  $P$ -faithful.  $\square$

In particular, the following groups afford projective representations which are irreducible and  $P$ -faithful:

- $\mathbf{Z}^n$  for any  $n \geq 2$ , as  $\mathbf{Z}^{n-1}$  is a dense subgroup of  $X(\mathbf{Z}) \approx \mathbf{T}$ .
- $\mathbf{Z}(p^\infty) \times \mathbf{Z}$ , as  $\mathbf{Z}$  is a dense subgroup of  $X(\mathbf{Z}(p^\infty)) \approx \mathbf{Z}_p$ . For the latter isomorphism, see e.g. [Bour–67], chap. 2, § 1, no. 9, cor. 4 of prop. 12;  $\mathbf{Z}(p^\infty)$  is as just after Proposition 18.
- $\mathbf{Q}^n$  for any  $n \geq 2$ . Indeed, let us check this for  $n = 2$ , the general case being entirely similar. The group  $X(\mathbf{Q})$  can be identified with  $\mathbf{A}/\varphi(\mathbf{Q})$ ; here  $\mathbf{A}$  is the group of adeles of  $\mathbf{Q}$  and  $\varphi : \mathbf{Q} \rightarrow \mathbf{A}$  is the diagonal embedding of  $\mathbf{Q}$  in  $\mathbf{A}$  (recall that  $\varphi(\mathbf{Q})$  is discrete and cocompact in  $\mathbf{A}$ ). More precisely, let  $\chi_0$  be a non-trivial character of  $\mathbf{A}$  with  $\chi_0|_{\varphi(\mathbf{Q})} = 1$ . Then the mapping

$$\Phi : \mathbf{A} \rightarrow X(\mathbf{Q}), \quad a \mapsto (q \mapsto \chi_0(a\varphi(q)))$$

factorizes to an isomorphism  $\mathbf{A}/\varphi(\mathbf{Q}) \rightarrow X(\mathbf{Q})$  (see Chapter 3 in [GGPS–90]). Fix  $a_0 \in \mathbf{A}$  with  $a_0 \notin \varphi(\mathbf{Q})$  and define a group homomorphism

$$f : \mathbf{Q} \rightarrow X(\mathbf{Q}), \quad f(q) = \Phi(a_0\varphi(q)).$$

Then  $f$  is injective since  $a_0\varphi(q) \notin \varphi(\mathbf{Q})$  for all  $q \in \mathbf{Q}^*$ . We claim that the range of  $f$  is dense. Indeed, assume that this is not the case. By Pontryagin duality, there exists  $q_0 \in \mathbf{Q}^*$  such that  $f(q)(\varphi(q_0)) = 1$  for all  $q \in \mathbf{Q}$ . This means that  $\chi_0(a_0\varphi(q_0q)) = 1$  for all  $q \in \mathbf{Q}$ , that is,  $\Phi(a_0\varphi(q_0))$  is the trivial character of  $\mathbf{Q}$ . This is a contradiction, since  $a_0\varphi(q_0) \notin \varphi(\mathbf{Q})$ .

Note that Proposition 19 carries over to dense subgroups of groups of the form  $B \times X(B)$ , with  $B$  a locally compact abelian group.

The case of finite groups is covered by a result of Frucht [Fruc–31]. For a modern exposition (and improvements<sup>9</sup>), see Page 166 of [BeZh–98].

**Proposition 20** (Frucht). *For a finite abelian group  $\Gamma$ , the two following properties are equivalent:*

- (i)  $\Gamma$  affords a projective representation which is irreducible and  $P$ -faithful;
- (ii) there exists a (finite abelian) group  $L$  such that  $\Gamma$  is isomorphic to the direct sum  $L \times L$ .

---

<sup>9</sup>Any finite abelian group affords two irreducible projective representations of which the direct sum is  $P$ -faithful. For a characterisation of those finite groups which have a faithful linear representation which is a direct sum of  $k$  irreducible representations, see Page 245, and indeed all of Chapter 9, in [BeZh–98].

*Observation*, from [Sury–08]. Consider a prime  $p$ , the “Heisenberg group”  $H$  below, and its noncyclic centre  $Z(H)$ :

$$H = \begin{pmatrix} 1 & \mathbf{F}_p & \mathbf{F}_{p^2} \\ 0 & 1 & \mathbf{F}_{p^2} \\ 0 & 0 & 1 \end{pmatrix} \supset Z(H) = \begin{pmatrix} 1 & 0 & \mathbf{F}_{p^2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \approx \mathbf{F}_{p^2} \approx \mathbf{F}_p \oplus \mathbf{F}_p$$

(where the last  $\approx$  indicates of course an isomorphism of additive groups, not of rings!). The quotient  $H/Z(H) \approx \mathbf{F}_p \oplus \mathbf{F}_{p^2}$  is an abelian group of which the order  $p$  is not a square, and therefore which does not have the properties of Proposition 20, but which is however a capable group.

Recall (from just after Proposition 19) that  $\mathbf{Z}^3$  affords a projective representation which is irreducible and P-faithful, and compare with Claim (ii) of Proposition 20.

## 9. Final remarks

In some sense, what follows goes back for finite groups to papers by Schur, from 1904 and 1907. For the general case, see Sections V.5 and V.6 in [Stam–73], [EcHS–72], and [Kerv–70].

A *stem cover* of a group  $\Gamma$  is a group  $\tilde{\Gamma}$  given with a surjection  $p$  onto  $\Gamma$  such that  $\ker(p)$  is central in  $\tilde{\Gamma}$ , contained in  $[\tilde{\Gamma}, \tilde{\Gamma}]$ , and isomorphic to  $H_2(\Gamma, \mathbf{Z})$ . Any group has a stem cover. The isomorphism type of  $\tilde{\Gamma}$  is uniquely determined in case  $\Gamma$  is perfect, but not in general. For example, the dihedral group of order 8 and the quaternion group both qualify for  $\tilde{\Gamma}$  if  $\Gamma$  is the Vierergruppe.

To check the existence of stem covers, consider  $H_2(\Gamma) := H_2(\Gamma, \mathbf{Z})$  as a trivial  $\Gamma$ -module, the short exact sequence

$$\{0\} \rightarrow \text{Ext}(\Gamma^{\text{ab}}, H_2(\Gamma)) \rightarrow H^2(\Gamma, H_2(\Gamma)) \rightarrow \text{Hom}(H_2(\Gamma), H_2(\Gamma)) \rightarrow \{0\}$$

of the universal coefficient theorem in cohomology, and a multiplier  $\zeta$  in  $Z^2(\Gamma, H_2(\Gamma))$  of which the cohomology class  $\underline{\zeta}$  is mapped onto the identity homomorphism of  $H_2(\Gamma)$  to itself. Then the corresponding central extension

$$\{0\} \longrightarrow H_2(\Gamma) \longrightarrow \tilde{\Gamma} \xrightarrow{p} \Gamma \longrightarrow \{1\},$$

in other words the central extension of characteristic class  $\underline{\zeta}$ , is a stem cover of  $\Gamma$ . Stem covers of  $\Gamma$  are classified (as central extensions of  $\Gamma$  by  $H_2(\Gamma)$ ) by the group  $\text{Ext}(\Gamma^{\text{ab}}, H_2(\Gamma, \mathbf{Z}))$ ; see Proposition V.5.3 of [Stam–73], and Theorem 2.2 of [EcHS–72]. In particular, if  $\Gamma$  is perfect, it has a unique stem cover, also called its *universal central extension*. If  $\Gamma$  is finite, its stem covers are also called its *Schur representation groups*.



Let  $p : \tilde{\Gamma} \longrightarrow \Gamma$  be a stem cover. For any central extension

$$\{0\} \longrightarrow A \longrightarrow \tilde{\Delta} \xrightarrow{q} \Delta \longrightarrow \{1\}$$

with divisible kernel  $A$  (more generally with  $A$  such that  $\text{Ext}(\Gamma^{\text{ab}}, A) = \{0\}$ ) and for any homomorphism  $\underline{\rho} : \Gamma \longrightarrow \Delta$ , there exists a homomorphism  $\rho^0 : \tilde{\Gamma} \longrightarrow \tilde{\Delta}$  such that  $\underline{\rho}(p(\tilde{\gamma})) = q(\rho^0(\tilde{\gamma}))$  for all  $\tilde{\gamma} \in \tilde{\Gamma}$ ; see Proposition V.5.5 of [Stam–73]. In particular, for a Hilbert space  $\mathcal{H}$  and a homomorphism  $\underline{\pi} : \Gamma \longrightarrow \mathcal{PU}(\mathcal{H})$ , there exists a unitary representation  $\pi^0 : \tilde{\Gamma} \longrightarrow \mathcal{U}(\mathcal{H})$  such that  $\underline{\pi}(p(\tilde{\gamma})) = p_{\mathcal{H}}(\pi^0(\tilde{\gamma}))$  for all  $\tilde{\gamma} \in \tilde{\Gamma}$ .

Observe that, if  $\Gamma$  is countable,  $H_2(\Gamma)$  is countable (this follows for example from the Schur-Hopf formula  $H_2(\Gamma) = R \cap [F, F]/[F, R]$  where  $\Gamma = F/R$  with  $F$  free), so that  $\tilde{\Gamma}$  is also countable.

It would be interesting to understand, say for the proof of Theorem 4, if and how one could use the stem cover(s) of  $\Gamma$  instead of the groups  $\Gamma(\zeta)$  which appear in Section 4.

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